13 D.c. circuit theory

At the end of this chapter you should be able to:

- state and use Kirchhoff's laws to determine unknown currents and voltages in d.c. circuits
- understand the superposition theorem and apply it to find currents in d.c. circuits
- understand general d.c. circuit theory
- understand Thévenin's theorem and apply a procedure to determine unknown currents in d.c. circuits
- recognize the circuit diagram symbols for ideal voltage and current sources
- understand Norton's theorem and apply a procedure to determine unknown currents in d.c. circuits
- appreciate and use the equivalence of the Thévenin and Norton equivalent networks
- state the maximum power transfer theorem and use it to determine maximum power in a d.c. circuit

13.1 Introduction

The laws which determine the currents and voltage drops in d.c. networks are: (a) Ohm's law (see Chapter 2), (b) the laws for resistors in series and in parallel (see Chapter 5), and (c) Kirchhoff's laws (see Section 13.2 following). In addition, there are a number of circuit theorems which have been developed for solving problems in electrical networks. These include:

(i) the superposition theorem (see Section 13.3),
(ii) Thévenin's theorem (see Section 13.5),
(iii) Norton's theorem (see Section 13.7), and
(iv) the maximum power transfer theorem (see Section 13.8).

13.2 Kirchhoff's laws

Kirchhoff's laws state:

(a) **Current Law.** *At any junction in an electric circuit the total current flowing towards that junction is equal to the total current flowing away from the junction*, i.e. $\Sigma I = 0$
Thus, referring to Figure 13.1:

$$I_1 + I_2 = I_3 + I_4 + I_5 \quad \text{or} \quad I_1 + I_2 - I_3 - I_4 - I_5 = 0$$

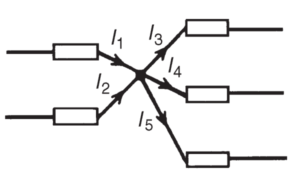


Figure 13.1

(b) **Voltage Law.** *In any closed loop in a network, the algebraic sum of the voltage drops (i.e. products of current and resistance) taken*

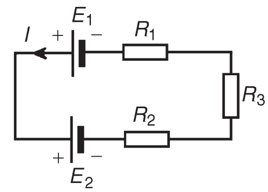


Figure 13.2

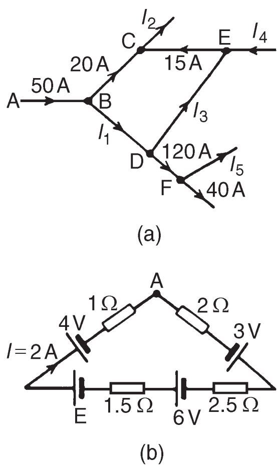


Figure 13.3

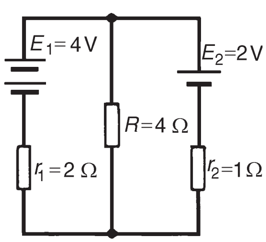


Figure 13.4

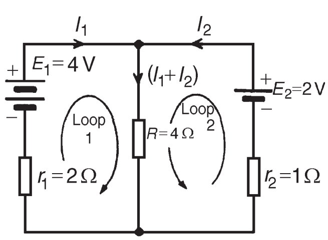


Figure 13.5

around the loop is equal to the resultant e.m.f. acting in that loop.

Thus, referring to Figure 13.2: $E_1 - E_2 = IR_1 + IR_2 + IR_3$

(Note that if current flows away from the positive terminal of a source, that source is considered by convention to be positive. Thus moving anticlockwise around the loop of Figure 13.2, E_1 is positive and E_2 is negative.)

> Problem 1. (a) Find the unknown currents marked in Figure 13.3(a). (b) Determine the value of e.m.f. E in Figure 13.3(b).

(a) Applying Kirchhoff's current law:

For junction B: $50 = 20 + I_1$. Hence $\boldsymbol{I_1 = 30}$ **A**

For junction C: $20 + 15 = I_2$. Hence $\boldsymbol{I_2 = 35}$ **A**

For junction D: $I_1 = I_3 + 120$

i.e. $30 = I_3 + 120$. Hence $\boldsymbol{I_3 = -90}$ **A**

(i.e. in the opposite direction to that shown in Figure 13.3(a))

For junction E: $I_4 + I_3 = 15$

i.e. $I_4 = 15 - (-90)$. Hence $\boldsymbol{I_4 = 105}$ **A**

For junction F: $120 = I_5 + 40$. Hence $\boldsymbol{I_5 = 80}$ **A**

(b) Applying Kirchhoff's voltage law and moving clockwise around the loop of Figure 13.3(b) starting at point A:

$$3 + 6 + E - 4 = (I)(2) + (I)(2.5) + (I)(1.5) + (I)(1)$$
$$= I(2 + 2.5 + 1.5 + 1)$$

i.e. $5 + E = 2(7)$, since $I = 2$ A

Hence $E = 14 - 5 = \mathbf{9\ V}$

> Problem 2. Use Kirchhoff's laws to determine the currents flowing in each branch of the network shown in Figure 13.4.

Procedure

1 Use Kirchhoff's current law and label current directions on the original circuit diagram. The directions chosen are arbitrary, but it is usual, as a starting point, to assume that current flows from the positive terminals of the batteries. This is shown in Figure 13.5 where the three branch currents are expressed in terms of I_1 and I_2 only, since the current through R is $I_1 + I_2$.

2 Divide the circuit into two loops and apply Kirchhoff's voltage law to each. From loop 1 of Figure 13.5, and moving in a clockwise direction

as indicated (the direction chosen does not matter), gives

$$E_1 = I_1 r_1 + (I_1 + I_2)R, \text{ i.e. } 4 = 2I_1 + 4(I_1 + I_2),$$

$$\text{i.e. } 6I_1 + 4I_2 = 4 \qquad (1)$$

From loop 2 of Figure 13.5, and moving in an anticlockwise direction as indicated (once again, the choice of direction does not matter; it does not have to be in the same direction as that chosen for the first loop), gives:

$$E_2 = I_2 r_2 + (I_1 + I_2)R, \text{ i.e. } 2 = I_2 + 4(I_1 + I_2),$$

$$\text{i.e. } 4I_1 + 5I_2 = 2 \qquad (2)$$

3 Solve equations (1) and (2) for I_1 and I_2.

$2 \times (1)$ gives: $12I_1 + 8I_2 = 8$ (3)

$3 \times (2)$ gives: $12I_1 + 15I_2 = 6$ (4)

$(3) - (4)$ gives: $-7I_2 = 2$ hence $I_2 = -\frac{2}{7} = \mathbf{-0.286\ A}$

(i.e. I_2 is flowing in the opposite direction to that shown in Figure 13.5.)

From (1) $6I_1 + 4(-0.286) = 4$

$$6I_1 = 4 + 1.144$$

Hence $I_1 = \frac{5.144}{6} = \mathbf{0.857\ A}$

Current flowing through resistance R is

$$I_1 + I_2 = 0.857 + (-0.286) = \mathbf{0.571\ A}$$

Note that a third loop is possible, as shown in Figure 13.6, giving a third equation which can be used as a check:

$$E_1 - E_2 = I_1 r_1 - I_2 r_2$$

$$4 - 2 = 2I_1 - I_2$$

$$2 = 2I_1 - I_2$$

[Check: $2I_1 - I_2 = 2(0.857) - (-0.286) = 2$]

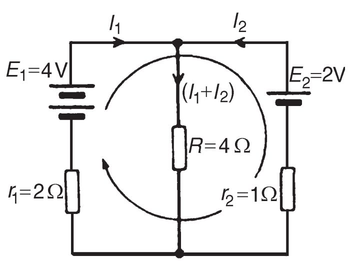


Figure 13.6

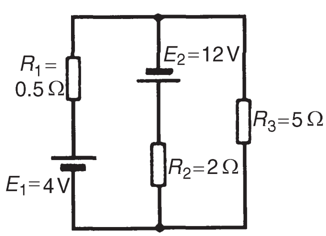


Figure 13.7

> Problem 3. Determine, using Kirchhoff's laws, each branch current for the network shown in Figure 13.7.

1 Currents, and their directions are shown labelled in Figure 13.8 following Kirchhoff's current law. It is usual, although not essential,

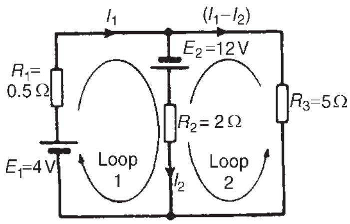


Figure 13.8

to follow conventional current flow with current flowing from the positive terminal of the source.

2 The network is divided into two loops as shown in Figure 13.8. Applying Kirchhoff's voltage law gives:

For loop 1:

$$E_1 + E_2 = I_1R_1 + I_2R_2$$

i.e. $\quad 16 = 0.5I_1 + 2I_2 \qquad (1)$

For loop 2:

$$E_2 = I_2R_2 - (I_1 - I_2)R_3$$

Note that since loop 2 is in the opposite direction to current$(I_1 - I_2)$, the volt drop across R_3 (i.e. $(I_1 - I_2)(R_3)$ is by convention negative).

Thus $\;12 = 2I_2 - 5(I_1 - I_2)$ i.e. $12 = -5I_1 + 7I_2 \qquad (2)$

3 Solving equations (1) and (2) to find I_1 and I_2:

$10 \times (1)$ gives $160 = 5I_1 + 20I_2 \qquad (3)$

$(2) + (3)$ gives $172 = 27I_2$ hence $I_2 = \dfrac{172}{27} = \mathbf{6.37\ A}$

From (1): $\quad 16 = 0.5I_1 + 2(6.37)$

$$I_1 = \frac{16 - 2(6.37)}{0.5} = \mathbf{6.52\ A}$$

Current flowing in $\boldsymbol{R_3} = I_1 - I_2 = 6.52 - 6.37 = \mathbf{0.15\ A}$

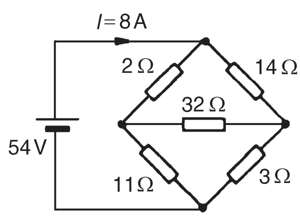


Figure 13.9

Problem 4. For the bridge network shown in Figure 13.9 determine the currents in each of the resistors.

Let the current in the 2 Ω resistor be I_1, then by Kirchhoff's current law, the current in the 14 Ω resistor is $(I - I_1)$. Let the current in the 32 Ω resistor be I_2 as shown in Figure 13.10. Then the current in the 11 Ω resistor is $(I_1 - I_2)$ and that in the 3 Ω resistor is $(I - I_1 + I_2)$. Applying Kirchhoff's voltage law to loop 1 and moving in a clockwise direction as shown in Figure 13.10 gives:

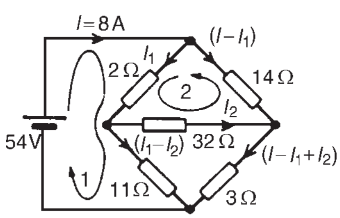


Figure 13.10

$$54 = 2I_1 + 11(I_1 - I_2)$$

i.e. $\quad 13I_1 - 11I_2 = 54 \qquad (1)$

Applying Kirchhoff's voltage law to loop 2 and moving in an anticlockwise direction as shown in Figure 13.10 gives:

$$0 = 2I_1 + 32I_2 - 14(I - I_1)$$

However $\quad I = 8$ A

Hence $\quad 0 = 2I_1 + 32I_2 - 14(8 - I_1)$

i.e. $\quad 16I_1 + 32I_2 = 112 \qquad (2)$

Equations (1) and (2) are simultaneous equations with two unknowns, I_1 and I_2.

16 × (1) gives: $\quad 208I_1 - 176I_2 = 864 \qquad (3)$

13 × (2) gives: $\quad 208I_1 + 416I_2 = 1456 \qquad (4)$

(4) − (3) gives: $\quad 592I_2 = 592$

$$I_2 = 1 \text{ A}$$

Substituting for I_2 in (1) gives:

$$13I_1 - 11 = 54$$

$$I_1 = \frac{65}{13} = 5\ A$$

Hence,
the current flowing in the 2 Ω resistor $= I_1 =$ **5 A**

the current flowing in the 14 Ω resistor $= I - I_1 = 8 - 5 =$ **3 A**

the current flowing in the 32 Ω resistor $= I_2 =$ **1 A**

the current flowing in the 11 Ω resistor $= I_1 - I_2 = 5 - 1 =$ **4 A** and

the current flowing in the 3 Ω resistor $= I - I_1 + I_2 = 8 - 5 + 1$
$=$ **4 A**

Further problems on Kirchhoff's laws may be found in Section 13.10, problems 1 to 6, page 189.

13.3 The superposition theorem

The **superposition theorem** states:

'In any network made up of linear resistances and containing more than one source of e.m.f., the resultant current flowing in any branch is the algebraic sum of the currents that would flow in that branch if each source was considered separately, all other sources being replaced at that time by their respective internal resistances.'

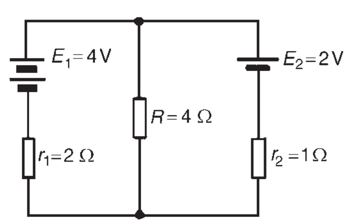


Figure 13.11

Problem 5. Figure 13.11 shows a circuit containing two sources of e.m.f., each with their internal resistance. Determine the current in each branch of the network by using the superposition theorem.

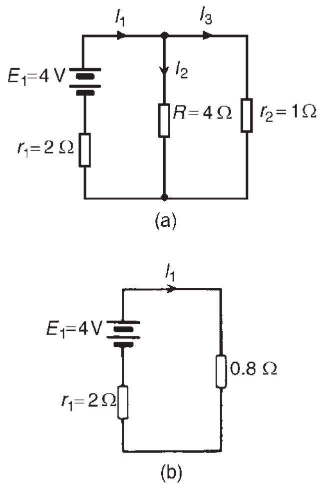


Figure 13.12

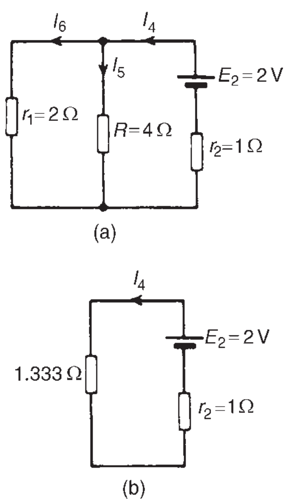


Figure 13.13

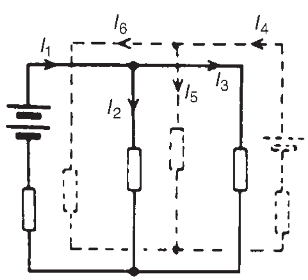


Figure 13.14

Procedure:

1 Redraw the original circuit with source E_2 removed, being replaced by r_2 only, as shown in Figure 13.12(a).

2 Label the currents in each branch and their directions as shown in Figure 13.12(a) and determine their values. (Note that the choice of current directions depends on the battery polarity, which, by convention is taken as flowing from the positive battery terminal as shown.) R in parallel with r_2 gives an equivalent resistance of:

$$\frac{4 \times 1}{4+1} = 0.8\ \Omega$$

From the equivalent circuit of Figure 13.12(b)

$$I_1 = \frac{E_1}{r_1 + 0.8} = \frac{4}{2+0.8} = 1.429\ \text{A}$$

From Figure 13.12(a)

$$I_2 = \left(\frac{1}{4+1}\right) I_1 = \frac{1}{5}(1.429) = 0.286\ \text{A}$$

and

$$I_3 = \left(\frac{4}{4+1}\right) I_1 = \frac{4}{5}(1.429) = 1.143\ \text{A by current division}$$

3 Redraw the original circuit with source E_1 removed, being replaced by r_1 only, as shown in Figure 13.13(a).

4 Label the currents in each branch and their directions as shown in Figure 13.13(a) and determine their values.
r_1 in parallel with R gives an equivalent resistance of:

$$\frac{2 \times 4}{2+4} = \frac{8}{6} = 1.333\ \Omega$$

From the equivalent circuit of Figure 13.13(b)

$$I_4 = \frac{E_2}{1.333 + r_2} = \frac{2}{1.333+1} = 0.857\ \text{A}$$

From Figure 13.13(a)

$$I_5 = \left(\frac{2}{2+4}\right) I_4 = \frac{2}{6}(0.857) = 0.286\ \text{A}$$

$$I_6 = \left(\frac{4}{2+4}\right) I_4 = \frac{4}{6}(0.857) = 0.571\ \text{A}$$

5 Superimpose Figure 13.13(a) on to Figure 13.12(a) as shown in Figure 13.14.

6 Determine the algebraic sum of the currents flowing in each branch. Resultant current flowing through source 1, i.e.

$$I_1 - I_6 = 1.429 - 0.571$$
$$= \mathbf{0.858\ A\ (discharging)}$$

Resultant current flowing through source 2, i.e.

$$I_4 - I_3 = 0.857 - 1.143$$
$$= \mathbf{-0.286\ A\ (charging)}$$

Resultant current flowing through resistor R, i.e.

$$I_2 + I_5 = 0.286 + 0.286$$
$$= \mathbf{0.572\ A}$$

The resultant currents with their directions are shown in Figure 13.15.

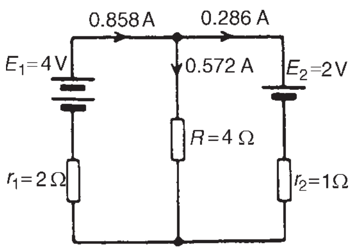


Figure 13.15

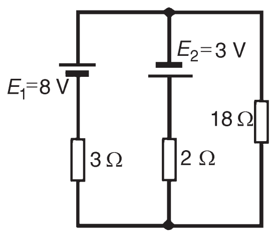


Figure 13.16

Problem 6. For the circuit shown in Figure 13.16, find, using the superposition theorem, (a) the current flowing in and the pd across the 18 Ω resistor, (b) the current in the 8 V battery and (c) the current in the 3 V battery.

1 Removing source E_2 gives the circuit of Figure 13.17(a).

2 The current directions are labelled as shown in Figure 13.17(a), I_1 flowing from the positive terminal of E_1.

From Figure 13.17(b), $I_1 = \dfrac{E_1}{3 + 1.8} = \dfrac{8}{4.8} = 1.667$ A

From Figure 13.17(a), $I_2 = \left(\dfrac{18}{2 + 18}\right) I_1 = \dfrac{18}{20}(1.667) = 1.500$ A

and $I_3 = \left(\dfrac{2}{2 + 18}\right) I_1 = \dfrac{2}{20}(1.667) = 0.167$ A

3 Removing source E_1 gives the circuit of Figure 13.18(a) (which is the same as Figure 13.18(b)).

4 The current directions are labelled as shown in Figures 13.18(a) and 13.18(b), I_4 flowing from the positive terminal of E_2

From Figure 13.18(c), $I_4 = \dfrac{E_2}{2 + 2.571} = \dfrac{3}{4.571} = 0.656$ A

From Figure 13.18(b), $I_5 = \left(\dfrac{18}{3 + 18}\right) I_4 = \dfrac{18}{21}(0.656) = 0.562$ A

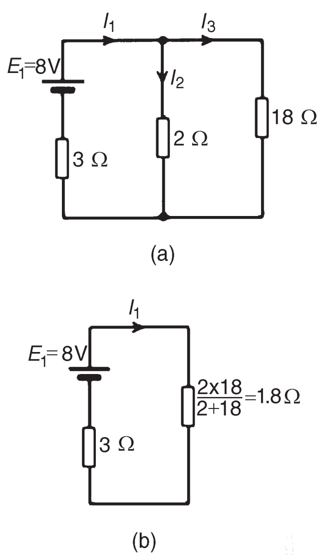


Figure 13.17

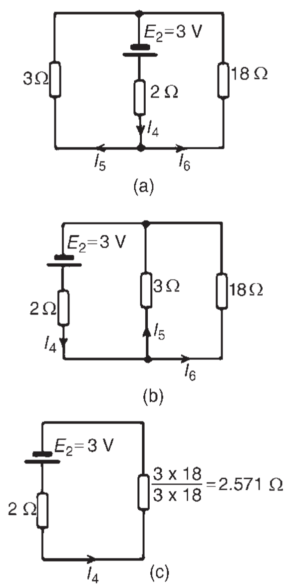


Figure 13.18

$$I_6 = \left(\frac{3}{3+18}\right) I_4 = \frac{3}{21}(0.656) = 0.094 \text{ A}$$

5 Superimposing Figure 13.18(a) on to Figure 13.17(a) gives the circuit in Figure 13.19.

6 (a) Resultant current in the 18 Ω resistor $= I_3 - I_6$

$= 0.167 - 0.094$

$=$ **0.073 A**

P.d. across the 18 Ω resistor = 0.073 × 18 = **1.314 V**

(b) Resultant current in th $= I_1 + I_5 = 1.667 + 0.562$

$=$ **2.229 A (discharging)**

(c) Resultant current in th $= I_2 + I_4 = 1.500 + 0.656$

$=$ **2.156 A (discharging)**

Further problems on the superposition theorem may be found in Section 13.10, problems 7 to 10, page 190.

13.4 General d.c. circuit theory

The following points involving d.c. circuit analysis need to be appreciated before proceeding with problems using Thévenin's and Norton's theorems:

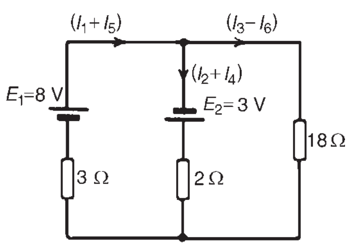


Figure 13.19

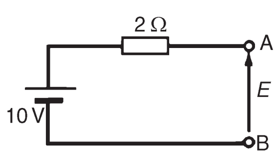


Figure 13.20

(i) The open-circuit voltage, E, across terminals AB in Figure 13.20 is equal to 10 V, since no current flows through the 2 Ω resistor and hence no voltage drop occurs.

(ii) The open-circuit voltage, E, across terminals AB in Figure 13.21(a) is the same as the voltage across the 6 Ω resistor. The circuit may be redrawn as shown in Figure 13.21(b).

$$E = \left(\frac{6}{6+4}\right)(50)$$

by voltage division in a series circuit, i.e. **E = 30 V**

(iii) For the circuit shown in Figure 13.22(a) representing a practical source supplying energy, $V = E - Ir$, where E is the battery e.m.f., V is the battery terminal voltage and r is the internal resistance of the battery (as shown in Section 4.6). For the circuit shown in Figure 13.22(b), $V = E - (-I)r$, i.e. $V = E + Ir$

(iv) The resistance 'looking-in' at terminals AB in Figure 13.23(a) is obtained by reducing the circuit in stages as shown in Figures 13.23(b) to (d). Hence the equivalent resistance across AB is 7 Ω

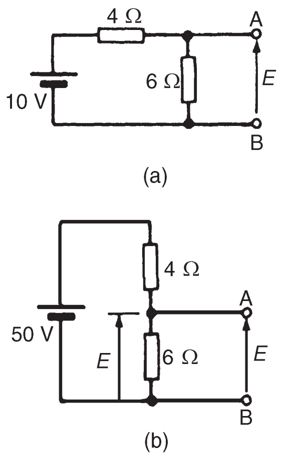


Figure 13.21

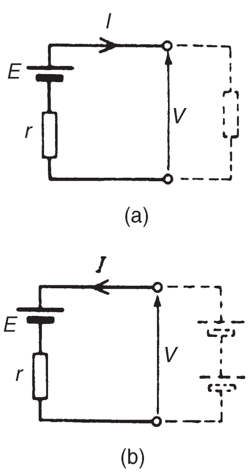


Figure 13.22

(v) For the circuit shown in Figure 13.24(a), the 3 Ω resistor carries no current and the p.d. across the 20 Ω resistor is 10 V. Redrawing the circuit gives Figure 13.24(b), from which

$$E = \left(\frac{4}{4+6}\right) \times 10 = \mathbf{4\ V}$$

(vi) If the 10 V battery in Figure 13.24(a) is removed and replaced by a short-circuit, as shown in Figure 13.24(c), then the 20 Ω resistor may be removed. The reason for this is that a short-circuit has zero resistance, and 20 Ω in parallel with zero ohms gives an equivalent resistance of: $(20 \times 0/20 + 0)$, i.e. 0 Ω. The circuit is then as shown in Figure 13.24(d), which is redrawn in Figure 13.24(e). From Figure 13.24(e), the equivalent resistance across AB,

$$r = \frac{6 \times 4}{6+4} + 3 = 2.4 + 3 = \mathbf{5.4\ \Omega}$$

(vii) To find the voltage across AB in Figure 13.25:
Since the 20 V supply is across the 5 Ω and 15 Ω resistors in series then, by voltage division, the voltage drop across AC,

$$V_{AC} = \left(\frac{5}{5+15}\right)(20) = 5\ \text{V}$$

Similarly, $V_{CB} = \left(\dfrac{12}{12+3}\right)(20) = 16$ V.

V_C is at a potential of +20 V.

$V_A = V_C - V_{AC} = +20 - 5 = 15\ V$ and

$V_B = V_C - V_{BC} = +20 - 16 = 4\ V$.

Hence the voltage between AB is $V_A - V_B = 15 - 4 = 11\ V$ and current would flow from A to B since A has a higher potential than B.

(viii) In Figure 13.26(a), to find the equivalent resistance across AB the circuit may be redrawn as in Figures 13.26(b) and (c). From Figure 13.26(c), the equivalent resistance across

$$\text{AB} = \frac{5 \times 15}{5+15} + \frac{12 \times 3}{12+3} = 3.75 + 2.4 = \mathbf{6.15\ \Omega}$$

(ix) In the worked problems in Sections 13.5 and 13.7 following, it may be considered that Thévenin's and Norton's theorems have

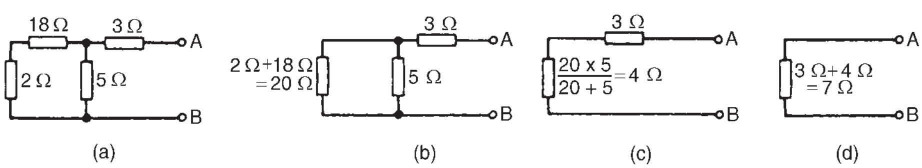


Figure 13.23

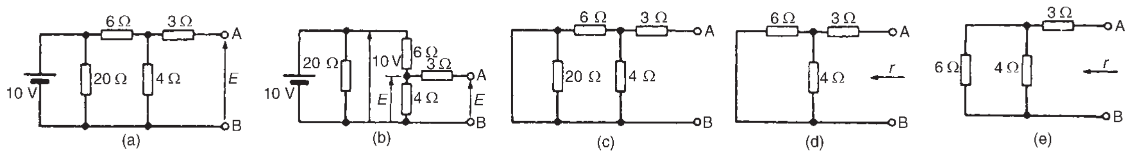


Figure 13.24

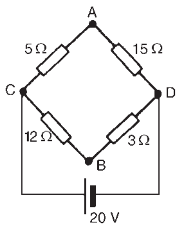


Figure 13.25

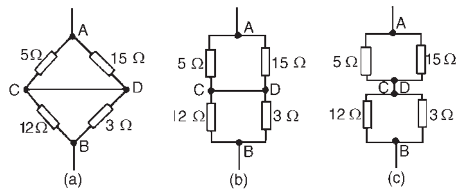


Figure 13.26

no obvious advantages compared with, say, Kirchhoff's laws. However, these theorems can be used to analyse part of a circuit and in much more complicated networks the principle of replacing the supply by a constant voltage source in series with a resistance (or impedance) is very useful.

13.5 Thévenin's theorem

Thévenin's theorem states:

'The current in any branch of a network is that which would result if an e.m.f. equal to the p.d. across a break made in the branch, were introduced into the branch, all other e.m.f.'s being removed and represented by the internal resistances of the sources.'

The procedure adopted when using Thévenin's theorem is summarized below. To determine the current in any branch of an active network (i.e. one containing a source of e.m.f.):

(i) remove the resistance R from that branch,
(ii) determine the open-circuit voltage, E, across the break,
(iii) remove each source of e.m.f. and replace them by their internal resistances and then determine the resistance, r, 'looking-in' at the break,
(iv) determine the value of the current from the equivalent circuit shown in Figure 13.27, i.e. $\boldsymbol{I = \dfrac{E}{R + r}}$

Problem 7. Use Thévenin's theorem to find the current flowing in the 10 Ω resistor for the circuit shown in Figure 13.28(a).

Following the above procedure:

(i) The 10 Ω resistance is removed from the circuit as shown in Figure 13.28(b)

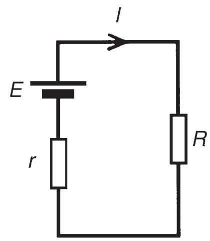


Figure 13.27

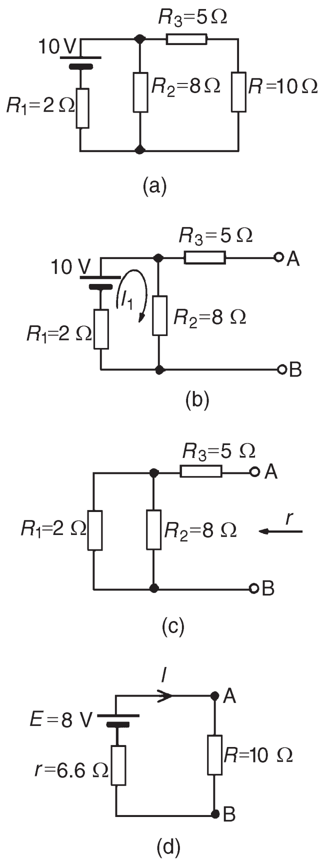


Figure 13.28

(ii) There is no current flowing in the 5 Ω resistor and current I_1 is given by:

$$I_1 = \frac{10}{R_1 + R_2} = \frac{10}{2 + 8} = 1 \text{ A}$$

P.d. across $R_2 = I_1 R_2 = 1 \times 8 = 8$ V

Hence p.d. across AB, i.e. the open-circuit voltage across the break, $E = 8$ V.

(iii) Removing the source of e.m.f. gives the circuit of Figure 13.28(c).

$$\text{Resistance, } r = R_3 + \frac{R_1 R_2}{R_1 + R_2} = 5 + \frac{2 \times 8}{2 + 8}$$

$$= 5 + 1.6 = 6.6 \ \Omega$$

(iv) The equivalent Thévenin's circuit is shown in Figure 13.28(d).

$$\text{Current } I = \frac{E}{R + r} = \frac{8}{10 + 6.6} = \frac{8}{16.6} = 0.482 \text{ A}$$

Hence the current flowing in the 10 Ω resistor of Figure 28(a) is **0.482 A**

> Problem 8. For the network shown in Figure 13.29(a) determine the current in the 0.8 Ω resistor using Thévenin's theorem.

Following the procedure:

(i) The 0.8 Ω resistor is removed from the circuit as shown in Figure 13.29(b).

(ii) $$\text{Current } I_1 = \frac{12}{1 + 5 + 4} = \frac{12}{10} = 1.2 \text{ A}$$

P.d. across 4 Ω resistor $= 4I_1 = (4)(1.2) = 4.8$ V

Hence p.d. across AB, i.e. the open-circuit voltage across AB, $E = 4.8$ V

(iii) Removing the source of e.m.f. gives the circuit shown in Figure 13.29(c). The equivalent circuit of Figure 13.29(c) is shown in Figure 13.29(d), from which,

$$\text{resistance } r = \frac{4 \times 6}{4 + 6} = \frac{24}{10} = 2.4 \ \Omega$$

(iv) The equivalent Thévenin's circuit is shown in Figure 13.29(e), from which,

$$\text{current } I = \frac{E}{r + R} = \frac{4.8}{2.4 + 0.8} + \frac{4.8}{3.2}$$

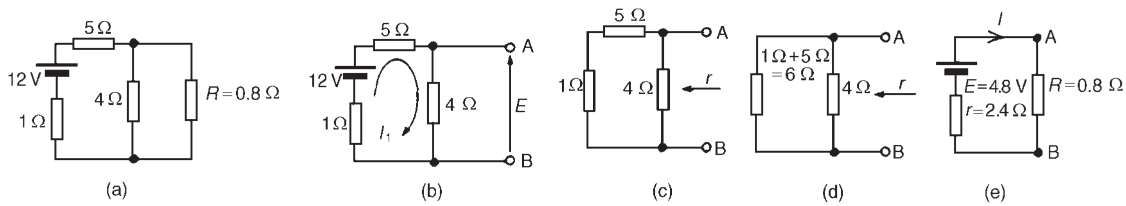


Figure 13.29

$$I = \mathbf{1.5\ A = current\ in\ the\ 0.8\ \Omega\ resistor}$$

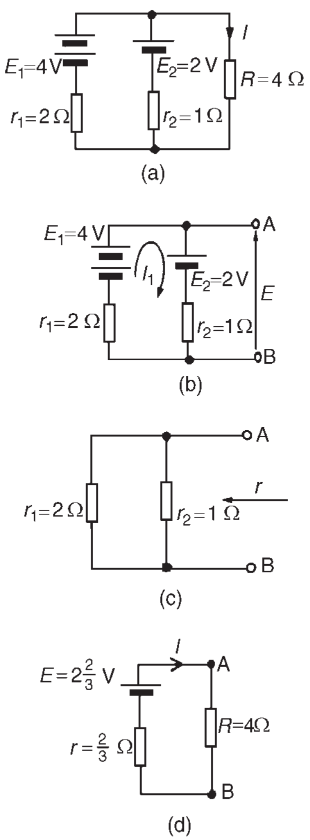


Figure 13.30

> Problem 9. Use Thévenin's theorem to determine the current I flowing in the 4 Ω resistor shown in Figure 13.30(a). Find also the power dissipated in the 4 Ω resistor.

Following the procedure:

(i) The 4 Ω resistor is removed from the circuit as shown in Figure 13.30(b).

(ii) Current $I_1 = \dfrac{E_1 - E_2}{r_1 + r_2} = \dfrac{4-2}{2+1} = \dfrac{2}{3}$ A

P.d. across AB, $E = E_1 - I_1 r_1 = 4 - \left(\frac{2}{3}\right)(2) = 2\frac{2}{3}$ V

(see Section 13.4(iii))

(Alternatively, p.d. across AB, $E = E_2 - I_1 r_2$

$= 2 - -\left(\frac{2}{3}\right)(1) = 2\frac{2}{3}$ V)

(iii) Removing the sources of e.m.f. gives the circuit shown in Figure 13.30(c), from which resistance

$$r = \frac{2 \times 1}{2+1} = \frac{2}{3}\ \Omega$$

(iv) The equivalent Thévenin's circuit is shown in Figure 13.30(d), from which,

$$\text{current, } I = \frac{E}{r+R} = \frac{2\frac{2}{3}}{\frac{2}{3}+4} = \frac{8/3}{14/3} = \frac{8}{14} = \mathbf{0.571\ A}$$

$$= \mathbf{current\ in\ the\ 4\ \Omega\ resistor}$$

Power dissipated in 4 Ω resistor, $P = I^2 R = (0.571)^2(4) = \mathbf{1.304\ W}$

> Problem 10. Use Thévenin's theorem to determine the current flowing in the 3 Ω resistance of the network shown in Figure 13.31(a). The voltage source has negligible internal resistance.

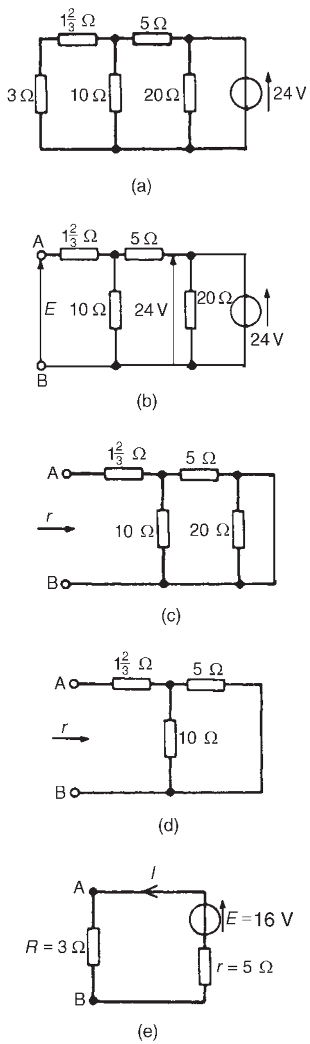


Figure 13.31

(Note the symbol for an ideal voltage source in Figure 13.31(a) which may be used as an alternative to the battery symbol.)

Following the procedure

(i) The 3 Ω resistance is removed from the circuit as shown in Figure 13.31(b).

(ii) The $1\frac{2}{3}\Omega$ resistance now carries no current.

$$\text{P.d. across } 10\ \Omega \text{ resistor} = \left(\frac{10}{10+5}\right)(24)$$

$$= \mathbf{16\ V} \text{ (see Section 13.4(v)).}$$

Hence p.d. across AB, $E = 16$ V

(iii) Removing the source of e.m.f. and replacing it by its internal resistance means that the 20 Ω resistance is short-circuited as shown in Figure 13.31(c) since its internal resistance is zero. The 20 Ω resistance may thus be removed as shown in Figure 13.31(d) (see Section 13.4 (vi)).

$$\text{From Figure 13.31(d), resistance, } r = 1\frac{2}{3} + \frac{10 \times 5}{10+5}$$

$$= 1\frac{2}{3} + \frac{50}{15} = 5\ \Omega$$

(iv) The equivalent Thévenin's circuit is shown in Figure 13.31(e), from which

$$\text{current, } I = \frac{E}{r+R} = \frac{16}{3+5} = \frac{16}{8} = \mathbf{2\ A}$$

$$= \textbf{current in the 3 Ω resistance}$$

Problem 11. A Wheatstone Bridge network is shown in Figure 13.32(a). Calculate the current flowing in the 32 Ω resistor, and its direction, using Thévenin's theorem. Assume the source of e.m.f. to have negligible resistance.

Following the procedure:

(i) The 32 Ω resistor is removed from the circuit as shown in Figure 13.32(b)

(ii) The p.d. between A and C, $V_{AC} = \left(\frac{R_1}{R_1 + R_4}\right)(E)$

$$= \left(\frac{2}{2+11}\right)(54) = 8.31\ \text{V}$$

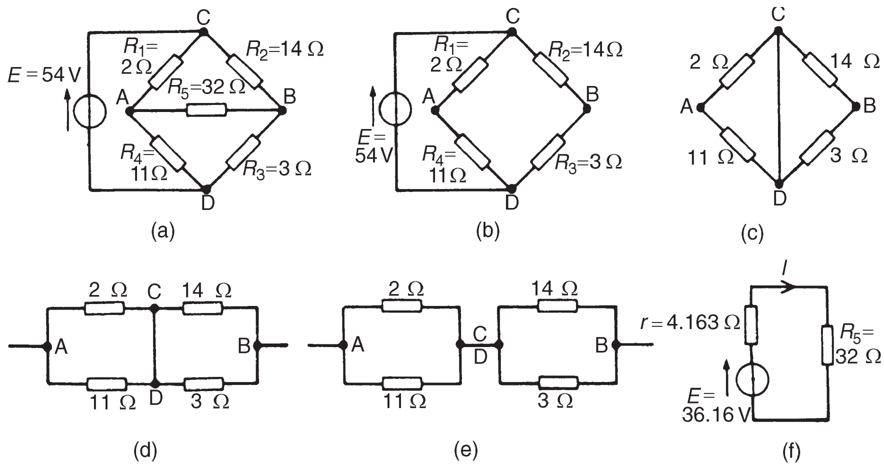


Figure 13.32

$$\text{The p.d. between B and C, } V_{BC} = \left(\frac{R_2}{R_2 + R_3}\right)(E)$$

$$= \left(\frac{14}{14+3}\right)(54) = 44.47 \text{ V}$$

Hence the p.d. between A and B = 44.47 – 8.31 = **36.16 V**
Point C is at a potential of + 54 V. Between C and A is a voltage drop of 8.31 V. Hence the voltage at point A is 54 – 8.31 = 45.69 V. Between C and B is a voltage drop of 44.47 V. Hence the voltage at point B is 54 – 44.47 = 9.53 V. Since the voltage at A is greater than at B, current must flow in the direction A to B. (See Section 13.4 (vii)).

(iii) Replacing the source of e.m.f. with a short-circuit (i.e. zero internal resistance) gives the circuit shown in Figure 13.32(c). The circuit is redrawn and simplified as shown in Figure 13.32(d) and (e), from which the resistance between terminals A and B,

$$r = \frac{2 \times 11}{2+11} + \frac{14 \times 3}{14+3} = \frac{22}{13} + \frac{42}{17} = 1.692 + 2.471 = \mathbf{4.163\ \Omega}$$

(iv) The equivalent Thévenin's circuit is shown in Figure 13.32(f), from which,

$$\text{current } I = \frac{E}{r + R_5} = \frac{36.16}{4.163 + 32} = 1 \text{ A}$$

Hence the current in the 32 Ω resistor of Figure 13.32(a) is 1 A, flowing from A to B

Further problems on Thévenin's theorem may be found in Section 13.10, problems 11 to 15, page 190.

13.6 Constant-current source

A source of electrical energy can be represented by a source of e.m.f. in series with a resistance. In Section 13.5, the Thévenin constant-voltage source consisted of a constant e.m.f. E in series with an internal resistance r. However this is not the only form of representation. A source of electrical energy can also be represented by a constant-current source in parallel with a resistance. It may be shown that the two forms are equivalent. An **ideal constant-voltage generator** is one with zero internal resistance so that it supplies the same voltage to all loads. An **ideal constant-current generator** is one with infinite internal resistance so that it supplies the same current to all loads.

Note the symbol for an ideal current source (BS 3939, 1985), shown in Figure 13.33.

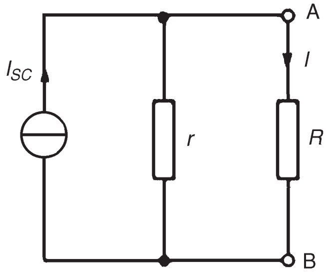


Figure 13.33

13.7 Norton's theorem

Norton's theorem states:

'The current that flows in any branch of a network is the same as that which would flow in the branch if it were connected across a source of electrical energy, the short-circuit current of which is equal to the current that would flow in a short-circuit across the branch, and the internal resistance of which is equal to the resistance which appears across the open-circuited branch terminals.'

The procedure adopted when using Norton's theorem is summarized below.

To determine the current flowing in a resistance R of a branch AB of an active network:

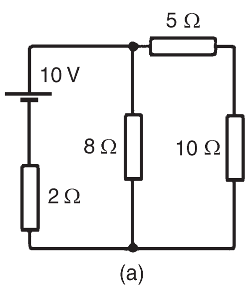
(i) short-circuit branch AB

(ii) determine the short-circuit current I_{SC} flowing in the branch

(iii) remove all sources of e.m.f. and replace them by their internal resistance (or, if a current source exists, replace with an open-circuit), then determine the resistance r,'looking-in' at a break made between A and B

(iv) determine the current I flowing in resistance R from the Norton equivalent network shown in Figure 13.33, i.e.

$$\boldsymbol{I = \left(\frac{r}{r+R}\right) I_{SC}}$$



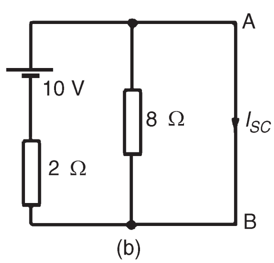


Figure 13.34

Problem 12. Use Norton's theorem to determine the current flowing in the 10 Ω resistance for the circuit shown in Figure 13.34(a).

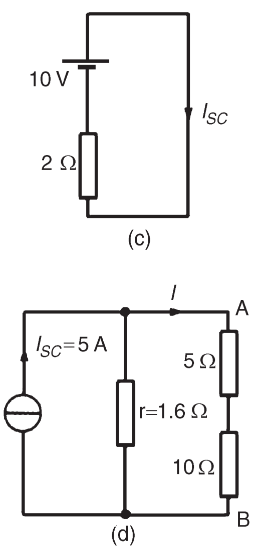


Figure 13.34 *continued*

Following the above procedure:

(i) The branch containing the 10 Ω resistance is short-circuited as shown in Figure 13.34(b).

(ii) Figure 13.34(c) is equivalent to Figure 13.34(b). Hence

$$I_{SC} = \tfrac{10}{2} = 5 \text{ A}$$

(iii) If the 10 V source of e.m.f. is removed from Figure 13.34(b) the resistance 'looking-in' at a break made between A and B is given by:

$$r = \frac{2 \times 8}{2 + 8} = 1.6 \ \Omega$$

(iv) From the Norton equivalent network shown in Figure 13.34(d) the current in the 10 Ω resistance, by current division, is given by:

$$I = \left(\frac{1.6}{1.6 + 5 + 10}\right)(5) = \mathbf{0.482 \ A}$$

as obtained previously in problem 7 using Thévenin's theorem.

Problem 13. Use Norton's theorem to determine the current I flowing in the 4 Ω resistance shown in Figure 13.35(a).

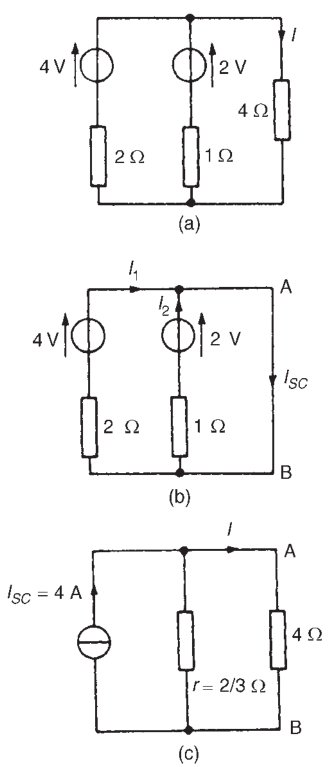


Figure 13.35

Following the procedure:

(i) The 4 Ω branch is short-circuited as shown in Figure 13.35(b).

(ii) From Figure 13.35(b), $I_{SC} = I_1 + I_2 = \frac{4}{2} + \frac{2}{1} = 4$ A

(iii) If the sources of e.m.f. are removed the resistance 'looking-in' at a break made between A and B is given by:

$$r = \frac{2 \times 1}{2 + 1} = \frac{2}{3} \ \Omega$$

(iv) From the Norton equivalent network shown in Figure 13.35(c) the current in the 4 Ω resistance is given by:

$$I = \left[\frac{2/3}{(2/3) + 4}\right](4) = \mathbf{0.571 \ A},$$

as obtained previously in problems 2, 5 and 9 using Kirchhoff's laws and the theorems of superposition and Thévenin.

Problem 14. Use Norton's theorem to determine the current flowing in the 3 Ω resistance of the network shown in Figure 13.36(a). The voltage source has negligible internal resistance.

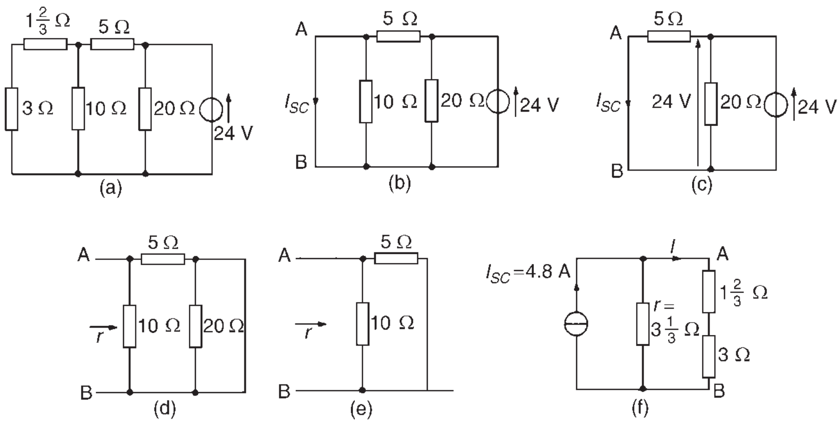


Figure 13.36

Following the procedure:

(i) The branch containing the 3 Ω resistance is short-circuited as shown in Figure 13.36(b).

(ii) From the equivalent circuit shown in Figure 13.36(c),

$$I_{SC} = \frac{24}{5} = 4.8 \text{ A}$$

(iii) If the 24 V source of e.m.f. is removed the resistance 'looking-in' at a break made between A and B is obtained from Figure 13.36(d) and its equivalent circuit shown in Figure 13.36(e) and is given by:

$$r = \frac{10 \times 5}{10 + 5} = \frac{50}{15} = 3\frac{1}{3}\Omega$$

(iv) From the Norton equivalent network shown in Figure 13.36(f) the current in the 3 Ω resistance is given by:

$$I = \left[\frac{3\frac{1}{3}}{3\frac{1}{3} + 1\frac{2}{3} + 3}\right](4.8) = \mathbf{2\ A,}$$

as obtained previously in problem 10 using Thévenin's theorem.

Problem 15. Determine the current flowing in the 2 Ω resistance in the network shown in Figure 13.37(a).

Following the procedure:

(i) The 2 Ω resistance branch is short-circuited as shown in Figure 13.37(b).

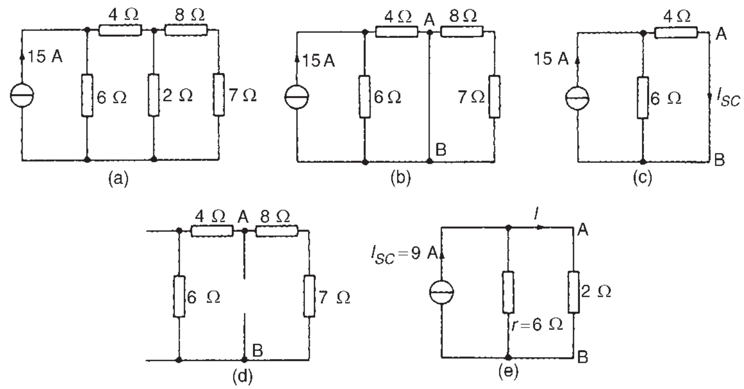


Figure 13.37

(ii) Figure 13.37(c) is equivalent to Figure 13.37(b).

Hence $I_{SC} = \left(\frac{6}{6+4}\right)(15) = \mathbf{9\ A}$ by current division.

(iii) If the 15 A current source is replaced by an open-circuit then from Figure 13.37(d) the resistance 'looking-in' at a break made between A and B is given by $(6 + 4)\ \Omega$ in parallel with $(8 + 7)\ \Omega$, i.e.

$$r = \frac{(10)(15)}{10 + 15} = \frac{150}{25} = 6\ \Omega$$

(iv) From the Norton equivalent network shown in Figure 13.37(e) the current in the $2\ \Omega$ resistance is given by:

$$I = \left(\frac{6}{6+2}\right)(9) = \mathbf{6.75\ A}$$

13.8 Thévenin and Norton equivalent networks

The Thévenin and Norton networks shown in Figure 13.38 are equivalent to each other. The resistance 'looking-in' at terminals AB is the same in each of the networks, i.e. r.

If terminals AB in Figure 13.38(a) are short-circuited, the short-circuit current is given by E/r. If terminals AB in Figure 13.38(b) are short-circuited, the short-circuit current is I_{SC}. For the circuit shown in Figure 13.38(a) to be equivalent to the circuit in Figure 13.38(b) the same short-circuit current must flow. Thus $I_{SC} = E/r$.

Figure 13.39 shows a source of e.m.f. E in series with a resistance r feeding a load resistance R.

From Figure 13.39, $I = \dfrac{E}{r+R} = \dfrac{E/r}{(r+R)/r} = \left(\dfrac{r}{r+R}\right)\dfrac{E}{r}$

$$\text{i.e. } I = \left(\frac{r}{r+R}\right) I_{SC}$$

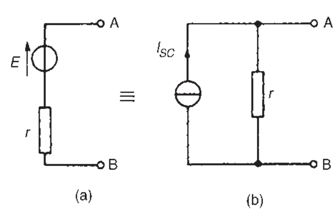


Figure 13.38

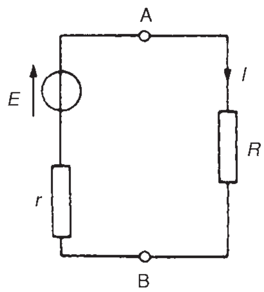


Figure 13.39

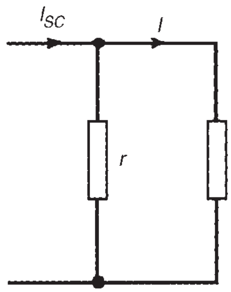


Figure 13.40

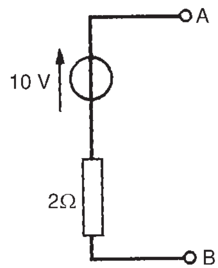


Figure 13.41

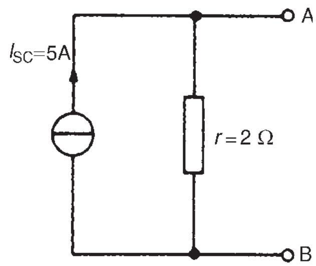


Figure 13.42

From Figure 13.40 it can be seen that, when viewed from the load, the source appears as a source of current I_{SC} which is divided between r and R connected in parallel.

Thus the two representations shown in Figure 13.38 are equivalent.

Problem 16. Convert the circuit shown in Figure 13.41 to an equivalent Norton network.

If terminals AB in Figure 13.41 are short-circuited, the short-circuit current $I_{SC} = \frac{10}{2} = 5$ A

The resistance 'looking-in' at terminals AB is 2 Ω. Hence the equivalent Norton network is as shown in Figure 13.42.

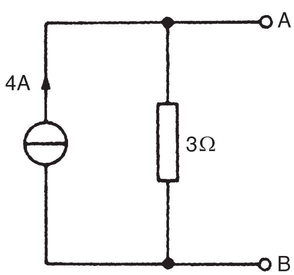


Figure 13.43

Problem 17. Convert the network shown in Figure 13.43 to an equivalent Thévenin circuit.

The open-circuit voltage E across terminals AB in Figure 13.43 is given by: $E = (I_{SC})(r) = (4)(3) = 12$ V.

The resistance 'looking-in' at terminals AB is 3 Ω. Hence the equivalent Thévenin circuit is as shown in Figure 13.44.

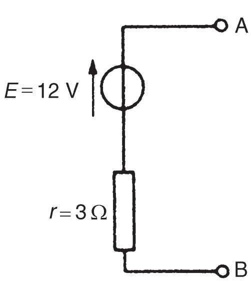


Figure 13.44

Problem 18. (a) Convert the circuit to the left of terminals AB in Figure 13.45(a) to an equivalent Thévenin circuit by initially converting to a Norton equivalent circuit. (b) Determine the current flowing in the 1.8 Ω resistor.

(a) For the branch containing the 12 V source, converting to a Norton equivalent circuit gives $I_{SC} = 12/3 = 4$ A and $r_1 = 3\Omega$. For the branch containing the 24 V source, converting to a Norton equivalent circuit gives $I_{SC2} = 24/2 = 12$ A and $r_2 = 2$ Ω.

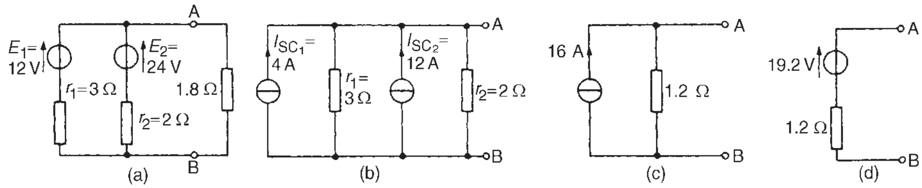


Figure 13.45

Thus Figure 13.45(b) shows a network equivalent to Figure 13.45(a). From Figure 13.45(b) the total short-circuit current is $4 + 12 = 16$ A

and the total resistance is given by: $\dfrac{3 \times 2}{3 + 2} = 1.2\ \Omega$

Thus Figure 13.45(b) simplifies to Figure 13.45(c).
The open-circuit voltage across AB of Figure 13.45(c),

$E = (16)(1.2) = 19.2$ V, and the resistance 'looking-in' at AB is 1.2 Ω. Hence the Thévenin equivalent circuit is as shown in Figure 13.45(d).

(b) When the 1.8 Ω resistance is connected between terminals A and B of Figure 13.45(d) the current I flowing is given by:

$$I = \frac{19.2}{1.2 + 1.8} = \mathbf{6.4\ A}$$

Problem 19. Determine by successive conversions between Thévenin and Norton equivalent networks a Thévenin equivalent circuit for terminals AB of Figure 13.46(a). Hence determine the current flowing in the 200 Ω resistance.

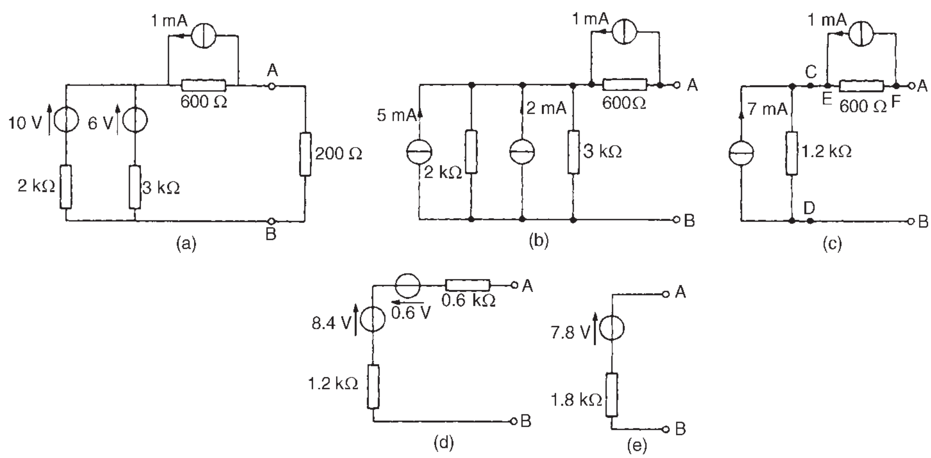


Figure 13.46

For the branch containing the 10 V source, converting to a Norton equivalent network gives

$$I_{SC} = \frac{10}{2000} = 5 \text{ mA and } r_1 = 2 \text{ k}\Omega.$$

6 V source, converting to a Norton equiv-

For the branch containing th
alent network gives

$$I_{SC} = \frac{6}{3000} = 2 \text{ mA and } r_2 = 3 \text{ k}\Omega.$$

Thus the network of Figure 13.46(a) converts to Figure 13.46(b).

Combining the 5 mA and 2 mA current sources gives the equivalent network of Figure 13.46(c) where the short-circuit current for the original two branches considered is 7 mA and the resistance is

$$\frac{2 \times 3}{2 + 3} = 1.2 \text{ k}\Omega.$$

Both of the Norton equivalent networks shown in Figure 13.46(c) may be converted to Thévenin equivalent circuits. The open-circuit voltage across CD is $(7 \times 10^{-3})(1.2 \times 10^3) = 8.4\ V$ and the resistance 'looking-in' at CD is 1.2 kΩ.

The open-circuit voltage across EF is $(1 \times 10^{-3})(600) = 0.6\ V$ and the resistance 'looking-in' at EF is 0.6 kΩ. Thus Figure 13.46(c) converts to Figure 13.46(d). Combining the two Thévenin circuits gives

$$E = 8.4 - 0.6 = \mathbf{7.8\ V} \text{ and the resistance}$$

$$r = (1.2 + 0.6) \text{ k}\Omega = \mathbf{1.8\ k\Omega}.$$

Thus the Thévenin equivalent circuit for terminals AB of Figure 13.46(a) is as shown in Figure 13.46(e).

Hence the current I flowing in a 200 Ω resistance connected between A and B is given by:

$$I = \frac{7.8}{1800 + 200} = \frac{7.8}{2000} = \mathbf{3.9\ mA}$$

Further problems on Norton's theorem may be found in Section 13.10, problems 16 to 21, page 191.

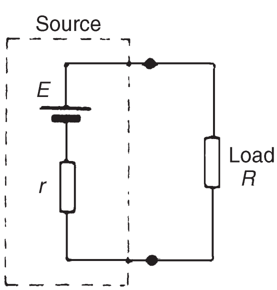


Figure 13.47

13.9 Maximum power transfer theorem

The **maximum power transfer theorem** states:

'The power transferred from a supply source to a load is at its maximum when the resistance of the load is equal to the internal resistance of the source.'

Hence, in Figure 13.47, when $R = r$ the power transferred from the source to the load is a maximum.

> Problem 20. The circuit diagram of Figure 13.48 shows dry cells of source e.m.f. 6 V, and internal resistance 2.5Ω. If the load resistance R_L is varied from 0 to 5 Ω in 0.5 Ω steps, calculate the power dissipated by the load in each case. Plot a graph of R_L (horizontally) against power (vertically) and determine the maximum power dissipated.

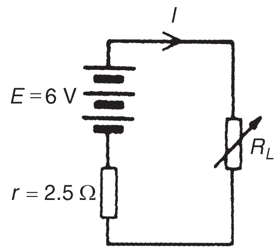


Figure 13.48

When $R_L = 0$, current $I = \dfrac{E}{r + R_L} = \dfrac{6}{2.5} = 2.4\ A$ and power dissipated in R_L, $P = I^2 R_L$, i.e. $P = (2.4)^2(0) = 0$ W

When $R_L = 0.5\ \Omega$, current $I = \dfrac{E}{r + R_L} = \dfrac{6}{2.5 + 0.5} = 2$ A

and $\quad P = I^2 R_L = (2)^2(0.5) = 2$ W

When $R_L = 1.0\Omega$, current $I = \dfrac{6}{2.5 + 1.0} = 1.714$ A

and $\quad P = (1.714)^2(1.0) = 2.94$ W

With similar calculations the following table is produced:

$R_L(\Omega)$	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$I = \dfrac{E}{r + R_L}$	2.4	2.0	1.714	1.5	1.333	1.2	1.091	1.0	0.923	0.857	0.8
$P = I^2 R_L (W)$	0	2.00	2.94	3.38	3.56	3.60	3.57	3.50	3.41	3.31	3.20

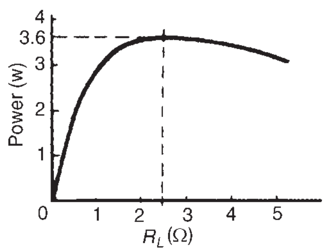


Figure 13.49

A graph of R_L against P is shown in Figure 13.49. **The maximum value of power is 3.60 W** which occurs when R_L is 2.5 Ω, i.e. **maximum power occurs when $R_L = r$**, which is what the maximum power transfer theorem states.

> Problem 21. A d.c. source has an open-circuit voltage of 30 V and an internal resistance of 1.5 Ω. State the value of load resistance that gives maximum power dissipation and determine the value of this power.

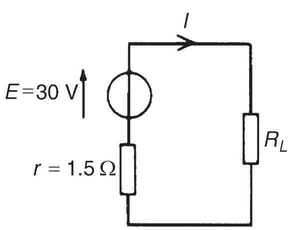


Figure 13.50

The circuit diagram is shown in Figure 13.50. From the maximum power transfer theorem, for maximum power dissipation,

$\boldsymbol{R_L = r = 1.5\ \Omega}$

From Figure 13.50, current $I = \dfrac{E}{r + R_L} = \dfrac{30}{1.5 + 1.5} = 10$ A

Power $P = I^2 R_L = (10)^2(1.5) =$ **150 W** = maximum power dissipated

Problem 22. Find the value of the load resistor R_L shown in Figure 13.51(a) that gives maximum power dissipation and determine the value of this power.

Using the procedure for Thévenin's theorem:

(i) Resistance R_L is removed from the circuit as shown in Figure 13.51(b).

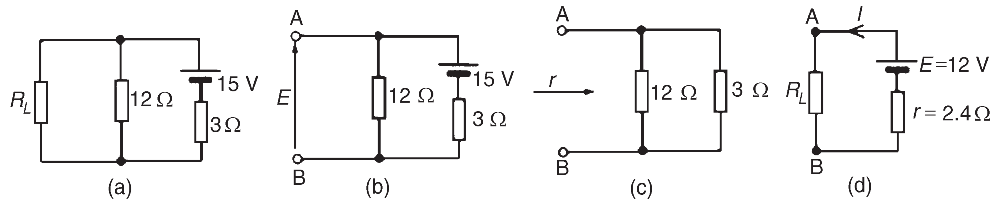


Figure 13.51

(ii) The p.d. across AB is the same as the p.d. across the 12 Ω resistor.

Hence $E = \left(\dfrac{12}{12+3}\right)(15) = 12$ V

(iii) Removing the source of e.m.f. gives the circuit of Figure 13.51(c),

from which resistance, $r = \dfrac{12 \times 3}{12+3} = \dfrac{36}{15} = 2.4\ \Omega$

(iv) The equivalent Thévenin's circuit supplying terminals AB is shown in Figure 13.51(d), from which, current, $I = E/(r + R_L)$

For maximum power, $R_L = r =$ **2.4 Ω**. Thus current,

$$I = \frac{12}{2.4 + 2.4} = 2.5 \text{ A.}$$

Power, P, dissipated in load R_L, $P = I^2 R_L = (2.5)^2(2.4) =$ **15 W**

Further problems on the maximum power transfer theorem may be found in Section 13.10 following, problems 22 and 23, page 192.

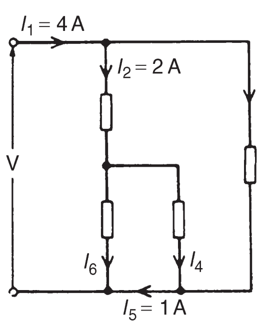


Figure 13.52

13.10 Further problems on d.c. circuit theory

Kirchhoff's laws

1 Find currents I_3, I_4 and I_6 in Figure 13.52

[$I_3 = 2$ A; $I_4 = -1$ A; $I_6 = 3$ A]

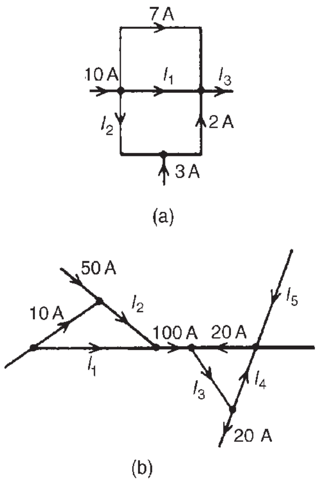


Figure 13.53

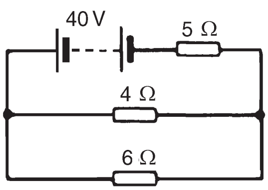


Figure 13.54

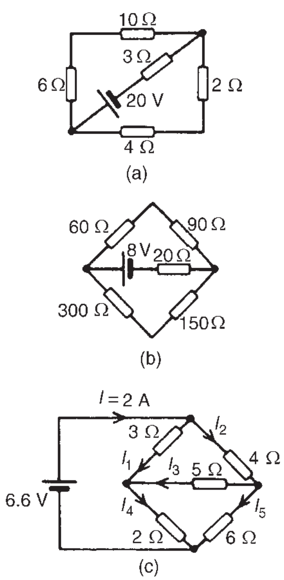


Figure 13.55

2 For the networks shown in Figure 13.53, find the values of the currents marked.
[(a) $I_1 = 4$ A, $I_2 = -1$ A, $I_3 = 13$ A
(b) $I_1 = 40$ A, $I_2 = 60$ A, $I_3 = 120$ A,
$I_4 = 100$ A, $I_5 = -80$ A]

3 Use Kirchhoff's laws to find the current flowing in the 6 Ω resistor of Figure 13.54 and the power dissipated in the 4 Ω resistor.
[2.162 A, 42.07 W]

4 Find the current flowing in the 3 Ω resistor for the network shown in Figure 13.55(a). Find also the p.d. across the 10 Ω and 2 Ω resistors.
[2.715 A, 7.410 V, 3.948 V]

5 For the networks shown in Figure 13.55(b) find: (a) the current in the battery, (b) the current in the 300 Ω resistor, (c) the current in the 90 Ω resistor, and (d) the power dissipated in the 150 Ω resistor.
[(a) 60.38 mA (b) 15.10 mA
(c) 45.28 mA (d) 34.20 mW]

6 For the bridge network shown in Figure 13.55(c), find the currents I_1 to I_5.
[$I_1 = 1.26$ A, $I_2 = 0.74$ A, $I_3 = 0.16$ A
$I_4 = 1.42$ A, $I_5 = 0.59$ A]

Superposition theorem

7 Use the superposition theorem to find currents I_1, I_2 and I_3 of Figure 13.56(a). [$I_1 = 2$ A, $I_2 = 3$ A, $I_3 = 5$ A]

8 Use the superposition theorem to find the current in the 8 Ω resistor of Figure 13.56(b). [0.385 A]

9 Use the superposition theorem to find the current in each branch of the network shown in Figure 13.56(c).
[10 V battery discharges at 1.429 A
4 V battery charges at 0.857 A
Current through 10 Ω resistor is 0.572 A]

10 Use the superposition theorem to determine the current in each branch of the arrangement shown in Figure 13.56(d).
[24 V battery charges at 1.664 A
52 V battery discharges at 3.280 A
Current in 20 Ω resistor is 1.616 A]

Thévenin's theorem

11 Use Thévenin's theorem to find the current flowing in the 14 Ω resistor of the network shown in Figure 13.57. Find also the power dissipated in the 14 Ω resistor. [0.434 A, 2.64 W]

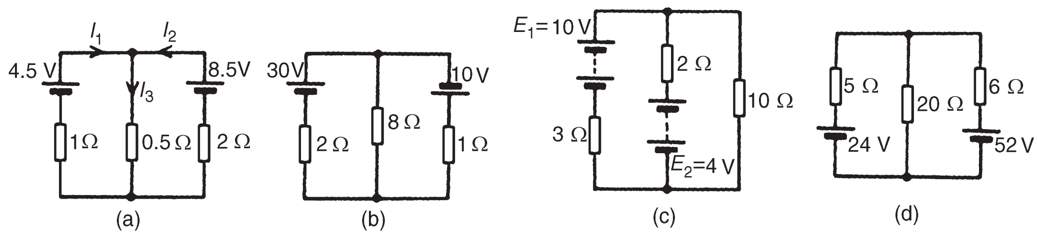


Figure 13.56

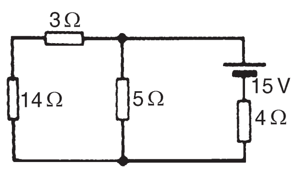


Figure 13.57

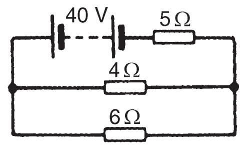


Figure 13.58

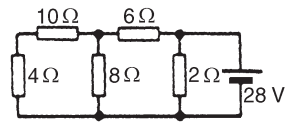


Figure 13.59

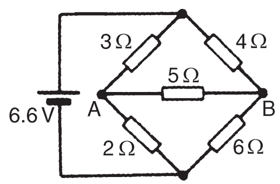


Figure 13.60

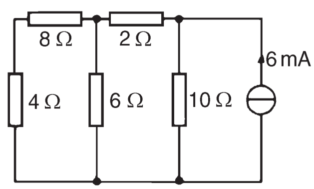


Figure 13.61

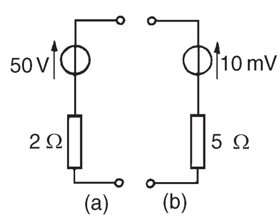


Figure 13.62

12 Use Thévenin's theorem to find the current flowing in the 6 Ω resistor shown in Figure 13.58 and the power dissipated in the 4 Ω resistor. [2.162 A, 42.07 W]

13 Repeat problems 7–10 using Thévenin's theorem.

14 In the network shown in Figure 13.59, the battery has negligible internal resistance. Find, using Thévenin's theorem, the current flowing in the 4 Ω resistor. [0.918 A]

15 For the bridge network shown in Figure 13.60, find the current in the 5 Ω resistor, and its direction, by using Thévenin's theorem.

[0.153 A from B to A]

Norton's theorem

16 Repeat problems 7–12, 14 and 15 using Norton's theorem.

17 Determine the current flowing in the 6 Ω resistance of the network shown in Figure 13.61 by using Norton's theorem. [2.5 mA]

18 Convert the circuits shown in Figure 13.62 to Norton equivalent networks.

[(a) $I_{SC} = 25$ A, $r = 2$ Ω
(b) $I_{SC} = 2$ mA, $r = 5$ Ω]

19 Convert the networks shown in Figure 13.63 to Thévenin equivalent circuits.

[(a) $E = 20$ V, $r = 4$ Ω
(b) $E = 12$ mV, $r = 3$ Ω]

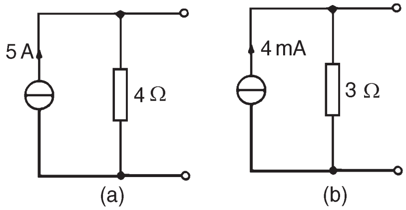


Figure 13.63

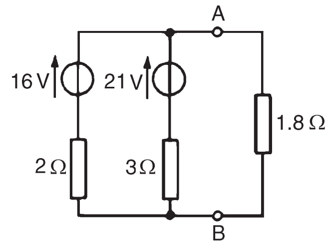


Figure 13.64

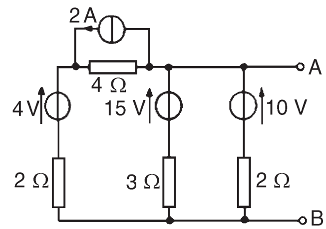


Figure 13.65

20 (a) Convert the network to the left of terminals AB in Figure 13.64 to an equivalent Thévenin circuit by initially converting to a Norton equivalent network.
(b) Determine the current flowing in the 1.8 Ω resistance connected between A and B in Figure 13.64.

[(a) $E = 18$ V, $r = 1.2$ Ω (b) 6 A]

21 Determine, by successive conversions between Thévenin and Norton equivalent networks, a Thévenin equivalent circuit for terminals AB of Figure 13.65. Hence determine the current flowing in a 6 Ω resistor connected between A and B.

[$E = 9\frac{1}{3}$ V, $r = 1$ Ω, $1\frac{1}{3}$ A]

Maximum power transfer theorem

22 A d.c. source has an open-circuit voltage of 20 V and an internal resistance of 2 Ω. Determine the value of the load resistance that gives maximum power dissipation. Find the value of this power.

[2 Ω, 50 W]

23 Determine the value of the load resistance R_L shown in Figure 13.66 that gives maximum power dissipation and find the value of the power.

[$R_L = 1.6$ Ω, $P = 57.6$ W]

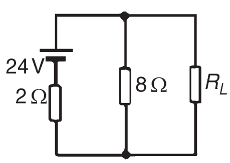


Figure 13.66